Chapter 1

Clifford algebras

This chapter presents what is a Clifford algebra, then we study the algebra of an Euclidean plane and the algebra of the three-dimensionnal physical space which is also the algebra of the Pauli matrices. We put there the space-time and the relativistic invariance. Then we present the space-time algebra and the Dirac matrices.

It is quite usual in a physics book to put into appendices mathematics even if they are necessary to understand the main part of the book. As it is impossible to expose the part containing physics without the Clifford algebras, we make here again a complete presentation of this necessary tool. [1]

We shall only speak here about Clifford algebras on the real field. Algebras on the complex field also exist and it could be expected to complex algebras to be key point for quantum physics. The main algebra used here is also an algebra on the complex field, but it is its structure of real algebra which is useful in this frame. [2]

Our aim is not to say everything about any Clifford algebra but simply to give to our lecturer tools to understand the next chapters of this book.

1. Readers being in the know may do a quick review. On the contrary a complete lecture is strongly advised for each reader who really wants to understand physics contained in the following chapters.
2. A Clifford algebra on the real field has components of vectors which are real numbers and which cannot be multiplied by i. A Clifford algebra on the complex field has components of vectors which are complex numbers and which can be multiplied by i.

1.1 What is a Clifford algebra?

1 - It is an algebra [12], there are two operations, noted $A+B$ and AB, such as, for any A, B, C :

$$\begin{aligned}
&A+(B+C)=(A+B)+C \quad ; \quad A+B=B+A\\
&A+0=A \quad ; \quad A+(-A)=0\\
&A(B+C)=AB+AC \quad ; \quad (A+B)C=AC+BC\\
&A(BC)=(AB)C.
\end{aligned} \tag{1.1}$$

2 - The algebra contains a set of vectors, noted with arrows, in which a scalar product exists and the intern Clifford multiplication $\vec{u}\vec{v}$ is supposed to verify for any vector $\vec{u}$:

$$\vec{u}\vec{u}=\vec{u}\cdot\vec{u}. \tag{1.2}$$

where $\vec{u}\cdot\vec{v}$ is the usual notation for the scalar product of two vectors. [3] This implies, as $\vec{u}\cdot\vec{u}$ is a real number, that the algebra contains vectors but also real numbers.

3 - Real numbers are commuting with any member of the algebra : if a is a real number and if A is any element in the algebra :

$$aA=Aa \tag{1.3}$$

$$1A=A. \tag{1.4}$$

Such an algebra exists for any finite-dimensionnal linear space which are the ones that we need in this book.

The smaller one is single, to within an isomorphism.

Remark 1 : (1.1) and (1.4) imply that the algebra is itself a linear space, not to be confused with the first one. If the initial linear space is n-dimensionnal, we get a Clifford algebra which is 2^n-dimensionnal. We shall see for instance in 1.3 that the Clifford algebra of the 3-dimensionnal physical space is a 8-dimensionnal linear space on the real field. We do not need here to distinguish the left or right linear space, as reals commute with each element of the algebra. We also do not need to consider the multiplication by a real as a third operation, because it is a particular case of the multiplication.

Remark 2 : If $\vec{u}$ and $\vec{v}$ are two orthogonal vectors, ($\vec{u}\cdot\vec{v}=0$), the equality $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})=(\vec{u}+\vec{v})(\vec{u}+\vec{v})$ implies $\vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}=\vec{u}\vec{u}+\vec{u}\vec{v}+\vec{v}\vec{u}+\vec{v}\vec{v}$, so we get :

$$0=\vec{u}\vec{v}+\vec{v}\vec{u} \quad ; \quad \vec{v}\vec{u}=-\vec{u}\vec{v} \tag{1.5}$$

3. This equality seems strange, but gives nice properties. We need these properties in the next chapters.

It's **the** change to usual rules on numbers, the multiplication is not commutative, we must be as careful as with matrix calculations.

Remark 3 : The addition is defined in the whole algebra, which contains numbers and vectors. So we shall get sums of numbers and vectors : $3+5\vec{i}$ is **authorized**. It is perhaps strange or disturbing, but we shall see next it is not different from $3+5i$. And everyone using complex numbers finally gets used to.

Even sub-algebra : It's the sub-algebra generated by the products of an even number of vectors : $\vec{u}\vec{v}$, $\vec{e}_1\vec{e}_2\vec{e}_3\vec{e}_4$,...

Reversion : The reversion $A \mapsto \widetilde{A}$ changes orders of products. Reversion does not change numbers a nor vectors $\tilde{a} = a$, $\tilde{\vec{u}} = \vec{u}$, and we get, for any $\vec{u}$ and $\vec{v}$, A and B :

$$\widetilde{\vec{u}\vec{v}} = \vec{v}\vec{u}\ ;\quad \widetilde{AB} = \tilde{B}\tilde{A}\ ;\quad \widetilde{A+B} = \tilde{A} + \tilde{B}. \tag{1.6}$$

1.2 Clifford algebra of an Euclidean plan : Cl_2

Cl_2 contains the real numbers and the vectors of an Euclidean plan, which read $\vec{u} = x\vec{e}_1 + y\vec{e}_2$, where $\vec{e}_1$ and $\vec{e}_2$ form an orthonormal basis of the plan : $\vec{e}_1{}^2 = \vec{e}_2{}^2 = 1$, $\vec{e}_1 \cdot \vec{e}_2 = 0$. Usually we set : $\vec{e}_1\vec{e}_2 = i$. The general element of the algebra is :

$$A = a + x\vec{e}_1 + y\vec{e}_2 + ib \tag{1.7}$$

where a, x, y and b are real numbers. This is enough because :

$$\begin{aligned}
\vec{e}_1 i &= \vec{e}_1(\vec{e}_1\vec{e}_2) = (\vec{e}_1\vec{e}_1)\vec{e}_2 = 1\vec{e}_2 = \vec{e}_2\\
\vec{e}_2 i &= -\vec{e}_1 \quad ; \quad i\vec{e}_2 = \vec{e}_1 \quad ; \quad i\vec{e}_1 = -\vec{e}_2\\
i^2 &= ii = i(\vec{e}_1\vec{e}_2) = (i\vec{e}_1)\vec{e}_2 = -\vec{e}_2\vec{e}_2 = -1
\end{aligned} \tag{1.8}$$

Two remarks must be made :

1- The even sub-algebra Cl_2^+ is the set formed by all $a+ib$, so it is the complex field. We may say that complex numbers are underlying as soon as the dimension of the linear space is greater than one. This even subalgebra is commutative.

2 - The reversion is the usual conjugation : $\tilde{i} = \widetilde{\vec{e}_1\vec{e}_2} = \vec{e}_2\vec{e}_1 = -i$

We get then, for any $\vec{u}$ and any $\vec{v}$ in the plane : $\vec{u}\vec{v} = \vec{u}\cdot\vec{v} + i\det(\vec{u},\vec{v})$

To establish that $(\vec{u}\cdot\vec{v})^2 + [\det(\vec{u},\vec{v})]^2 = \vec{u}^{\,2}\vec{v}^{\,2}$, it is possible to use $\vec{u}\vec{v}\vec{v}\vec{u}$ which can be calculated by two ways, and we can use $\vec{v}\vec{v}$ which is a real number and commutes with anything in the algebra.

1.3 Clifford algebra of the physical space : Cl_3

Cl_3 [2] contains the real numbers and the vectors of the physical space which read : $\vec{u} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$, where x, y and z are real numbers, $\vec{e}_1$, $\vec{e}_2$

and $\vec{e}_3$ form an orthonormal basis :

$$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_3 = \vec{e}_3 \cdot \vec{e}_1 = 0 \;\; ; \;\; \vec{e}_1{}^2 = \vec{e}_2{}^2 = \vec{e}_3{}^2 = 1. \qquad (1.9)$$

We let :

$$i_1 = \vec{e}_2\vec{e}_3 \;\; ; \;\; i_2 = \vec{e}_3\vec{e}_1 \;\; ; \;\; i_3 = \vec{e}_1\vec{e}_2 \;\; ; \;\; i = \vec{e}_1\vec{e}_2\vec{e}_3. \qquad (1.10)$$

Then we get :

$$i_1^2 = i_2^2 = i_3^2 = i^2 = -1 \qquad (1.11)$$

$$i\vec{u} = \vec{u}i \;\; ; \;\; i\vec{e}_j = i_j \;,\; j = 1,\; 2,\; 3. \qquad (1.12)$$

To verify (1.11) we can use the same way we used to get (1.8). To verify (1.12) we may firstly justify that i commutes with each $\vec{e}_j$.

The general element of Cl_3 reads : $A = a + \vec{u} + i\vec{v} + ib$. This gives $1+3+3+1 = 8 = 2^3$ dimensions for Cl_3.

Several remarks :

1 - The center of Cl_3 is the set of the $a+ib$, only elements which commute with every other ones in the algebra. It is isomorphic to the complex field.

2 - The even sub-algebra Cl_3^+ is the set of the $a+i\vec{v}$, isomorphic to the quaternion field. Therefore quaternions are implicitly present into calculations as soon as the dimension of the linear space is greater or equal to three. This even subalgebra is not commutative.

3 - $\tilde{A} = a + \vec{u} - i\vec{v} - ib$; The reversion is the conjugation, for complex numbers but also for the quaternions contained into Cl_3.

4 - $i\vec{v}$ is what is usually called "axial vector" or "pseudo-vector", whilst $\vec{u}$ is usually called vector. It is well known that it is specific to dimension 3.

5 - There are now four different terms with square -1, four ways to get complex numbers. Quantum theory is used to only one term with square -1. When complex numbers are used in quantum mechanics, it is necessary to ask the question of which i is used : i or i_3 ?

1.3.1 Cross-product, orientation

$\vec{u} \times \vec{v}$ is the cross-product of $\vec{u}$ and $\vec{v}$. Using coordinates in the basis $(\vec{e}_1,\, \vec{e}_2,\, \vec{e}_3)$, we can easily establish for any $\vec{u}$ and $\vec{v}$:

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + i\; \vec{u} \times \vec{v} \qquad (1.13)$$

$$(\vec{u} \cdot \vec{v})^2 + (\vec{u} \times \vec{v})^2 = \vec{u}^{\,2}\vec{v}^{\,2} \qquad (1.14)$$

$\det(\vec{u}, \vec{v}, \vec{w})$ is the determinant whose columns contain the components of vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, in the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Again using coordinates, it is possible to establish, for any $\vec{u}$, $\vec{v}$, $\vec{w}$:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det(\vec{u}, \vec{v}, \vec{w}) \qquad (1.15)$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{w} \cdot \vec{u})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \qquad (1.16)$$

$$\vec{u}\vec{v}\vec{w} = i \det(\vec{u}, \vec{v}, \vec{w}) + (\vec{v} \cdot \vec{w})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v} + (\vec{u} \cdot \vec{v})\vec{w} \qquad (1.17)$$

From (1.15) comes that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$. (1.14) allows to calculate the length of $\vec{u} \times \vec{v}$, and (1.15) gives its orientation. We recall that a basis $(\vec{u}, \vec{v}, \vec{w})$ is said to be direct, or to have the same orientation as $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ if $\det(\vec{u}, \vec{v}, \vec{w}) > 0$ and to be inverse, or to have other orientation if $\det(\vec{u}, \vec{v}, \vec{w}) < 0$. (1.17) allows to establish that, if $B = (\vec{u}, \vec{v}, \vec{w})$ is any orthonormal basis, then $\vec{u}\vec{v}\vec{w} = i$ if and only if B is direct, and $\vec{u}\vec{v}\vec{w} = -i$ if and only if B is inverse. So i is strictly linked to the orientation of the physical space. To change i to $-i$ is equivalent to change the space orientation (it is the same for a plan). The fact that i rules the orientation of the physical space will play an important role in the next chapters.

1.3.2 Pauli algebra

The Pauli algebra, introduced in physics as soon as 1926 to account for the spin of the electron, is the algebra of 2×2 complex matrices. It is identical (isomorphic) to Cl_3, but only as algebras on the real field.[4] Identifying complex numbers to scalar matrices and the e_j to the Pauli matrices σ_j is enough.[5] So we have

$$z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \tag{1.18}$$

$$\vec{e}_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ;\ \vec{e}_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ;\ \vec{e}_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.19}$$

This is fully compatible with the preceding calculations, because :

$$\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = i \tag{1.20}$$

$$\sigma_1\sigma_2 = i\sigma_3 \quad ; \quad \sigma_2\sigma_3 = i\sigma_1 \quad ; \quad \sigma_3\sigma_1 = i\sigma_2 \tag{1.21}$$

And reverse is identical to adjoint :

$$\tilde{A} = A^\dagger \tag{1.22}$$

Consequently we shall say now equally Pauli algebra or Cl_3. This is not liked by pure Clifford algebrists. As for physicists, they are used to Pauli algebra and to old and cumbersome notations, as they do not use conveniences of the Clifford algebra of the physical space : Cl_3.

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4. The dimension of the Pauli algebra is 8 on the real field, but only 4 on the complex field.

5. This identifying process may be considered a lack of rigor, but in fact it is frequent in mathematics. The same process allows to include integer numbers into relative numbers, or real numbers into complex numbers. To go without implies very complicated notations. This identifying process considers the three σ_j as forming a direct basis of the physical space.

1.3.3 Three conjugations will be useful :

$A = a+\vec{u}+i\vec{v}+ib$ is the sum of the even part $A_1 = a+i\vec{v}$ (quaternion) and the odd part $A_2 = \vec{u} + ib$. From this we define the conjugation (involutive automorphism) notated $\widehat{A}$ such as

$$\widehat{A} = A_1 - A_2 = a - \vec{u} + i\vec{v} - ib \tag{1.23}$$

This conjugation verifies, for any element A and any B in Cl_3 :

$$\widehat{A+B} = \widehat{A} + \widehat{B} \;\;;\;\; \widehat{AB} = \widehat{A}\widehat{B}. \tag{1.24}$$

From this conjugation and from the reversion we may form the third conjugation :

$$\overline{A} = \widehat{A}^{\dagger} = a - \vec{u} - i\vec{v} + ib \;\;:\;\; \overline{A+B} = \overline{A} + \overline{B} \;\;;\;\; \overline{AB} = \overline{B}\,\overline{A} \tag{1.25}$$

Composing, in any order, two of those three conjugations gives the third one. Only $A \mapsto \widehat{A}$ preserves the order of products, $A \mapsto \overline{A}$ and $A \mapsto A^{\dagger}$ inverse the order of products.

Now a, b, c, d are any complex numbers and a^* is the complex conjugate of a. We can verify that for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have :

$$\widetilde{A} = A^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \;\;;\;\; \widehat{A} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} \;\;;\;\; \overline{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{1.26}$$

$$A\overline{A} = \overline{A}A = \det(A) = ad - bc \;\;;\;\; \widehat{A}A^{\dagger} = A^{\dagger}\widehat{A} = [\det(A)]^* \tag{1.27}$$

1.3.4 Gradient, divergence and curl :

In Cl_3 there exists one important differential operator, because all may be made with it :

$$\vec{\partial} = \vec{e}_1\partial_1 + \vec{e}_2\partial_2 + \vec{e}_3\partial_3 = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix} \tag{1.28}$$

with [6]

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \;\;;\;\; \partial_j = \frac{\partial}{\partial x^j} \tag{1.29}$$

The Laplacian is simply the square of $\vec{\partial}$:

$$\Delta = (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2 = \vec{\partial}\vec{\partial} \tag{1.30}$$

6. This operator $\vec{\partial}$ is usually notated, in quantum mechanics, as a scalar product, for instance $\vec{\sigma} \cdot \vec{\nabla}$. From this come many convoluted complications. To use simple notations fully simplifies calculations.

Applied to a scalar a, $\vec{\partial}$ gives the gradient and applied to a vector $\vec{u}$ it gives both the divergence and the curl :

$$\vec{\partial}a = \vec{\text{grad}}\ a \tag{1.31}$$

$$\vec{\partial}\vec{u} = \vec{\partial}\cdot\vec{u} + i\ \vec{\partial}\times\vec{u} \ \ ; \ \ \vec{\partial}\cdot\vec{u} = \text{div}\vec{u} \ \ ; \ \ \vec{\partial}\times\vec{u} = \text{curl}\ \vec{u}. \tag{1.32}$$

1.3.5 Space-time in space algebra :

With

$$x^0 = ct \ \ ; \ \ \vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \ \ ; \ \ \partial_\mu = \frac{\partial}{\partial x^\mu} \tag{1.33}$$

we let [2][31]:

$$x = x^0 + \vec{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{1.34}$$

Then the space-time is made of the auto-adjoint part of the Pauli algebra ($x^\dagger = x$) and we get

$$\widehat{x} = \overline{x} = x^0 - \vec{x} \tag{1.35}$$

$$\det(x) = x\widehat{x} = x\cdot x = (x^0)^2 - \vec{x}^{\,2} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{1.36}$$

The square of the pseudo-norm of any space-time vector is then simply the determinant. [7] Any element M of the Pauli algebra is the sum of a space-time vector v and of the product by i of another space-time vector w :

$$M = v + iw \tag{1.37}$$

$$v = \frac{1}{2}(M + M^\dagger) \ ; \ \ v^\dagger = v \tag{1.38}$$

$$iw = \frac{1}{2}(M - M^\dagger) \ ; \ \ w^\dagger = w. \tag{1.39}$$

Space-time vectors v and w are uniquely defined. We need two linked differential operators :

$$\nabla = \partial_0 - \vec{\partial} \ \ ; \ \ \widehat{\nabla} = \partial_0 + \vec{\partial} \tag{1.40}$$

They allow to calculate the D'alembertian :

$$\nabla\widehat{\nabla} = \widehat{\nabla}\nabla = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2 = \Box \tag{1.41}$$

7. We must notice that the pseudo-norm of the space-time metric comes not from a scalar product, a symmetric bilinear form, but from a determinant, an antisymmetric bilinear form. We are here very far from Riemannian spaces.