16. Inflation-Indexed Swaps

Given a set of dates $T_1, \ldots, T_M$, an Inflation-Indexed Swap (IIS) is a swap where, on each payment date, Party A pays Party B the inflation rate over a predefined period, while Party B pays Party A a fixed rate. The inflation rate is calculated as the percentage return of the CPI index over the time interval it applies to. Two are the main IIS traded in the market: the zero coupon (ZC) swap and the year-on-year (YY) swap.

In a ZCIIS, at the final time T_M, assuming $T_M = M$ years, Party B pays Party A the fixed amount

$$N[(1+K)^M - 1], \tag{16.1}$$

where K and N are, respectively, the contract fixed rate and nominal value. In exchange for this fixed payment, Party A pays Party B, at the final time T_M, the floating amount

$$N\left[\frac{I(T_M)}{I_0} - 1\right]. \tag{16.2}$$

In a YYIIS, at each time T_i, Party B pays Party A the fixed amount

$$N\varphi_i K,$$

where φ_i is the contract fixed-leg year fraction for the interval $[T_{i-1}, T_i]$, while Party A pays Party B the (floating) amount

$$N\psi_i\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right], \tag{16.3}$$

where ψ_i is the floating-leg year fraction for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and N is again the contract nominal value.

Both ZC and YY swaps are quoted, in the market, in terms of the corresponding fixed rate K. The ZCIIS and YYIIS (mid) fixed-rate quotes in the Euro market on October 7th 2004 are shown in Figure 16.1, for maturities up to twenty years. The reference CPI is the Euro-zone ex-tobacco index.

16.1 Pricing of a ZCIIS

Standard no-arbitrage pricing theory implies that the value at time t, $0 \leq t < T_M$, of the inflation-indexed leg of the ZCIIS is

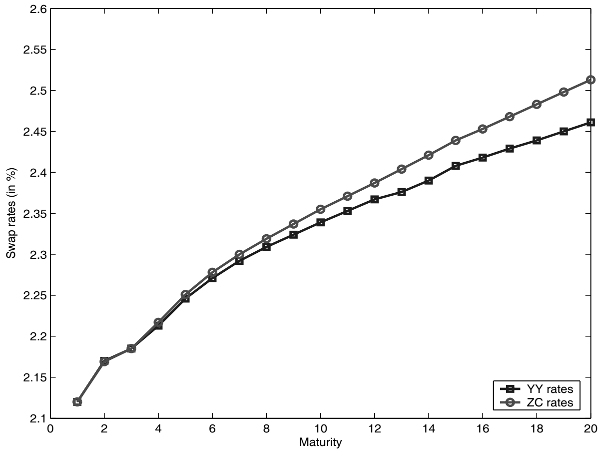

Fig. 16.1. Euro inflation swap rates as of October 7, 2004.

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = N E_n \left\{ e^{-\int_t^{T_M} n(u)\,du} \left[\frac{I(T_M)}{I_0} - 1 \right] \Big| \mathcal{F}_t \right\}, \tag{16.4}$$

where $\mathcal{F}_t$ denotes the σ-algebra generated by the relevant underlying processes up to time t.

By the foreign-currency analogy, the nominal price of a real zero-coupon bond equals the nominal price of the contract paying off one unit of the CPI index at bond maturity, see also the general formula (2.31). In formulas, for each $t < T$:

$$I(t) P_r(t, T) = I(t) E_r \left\{ e^{-\int_t^T r(u)\,du} \Big| \mathcal{F}_t \right\} = E_n \left\{ e^{-\int_t^T n(u)\,du} I(T) \Big| \mathcal{F}_t \right\}. \tag{16.5}$$

Therefore, (16.4) becomes

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = N \left[\frac{I(t)}{I_0} P_r(t, T_M) - P_n(t, T_M) \right], \tag{16.6}$$

which at time $t = 0$ simplifies to

$$\mathbf{ZCIIS}(0, T_M, N) = N [P_r(0, T_M) - P_n(0, T_M)]. \tag{16.7}$$

Formulas (16.6) and (16.7) yield model-independent prices, which are not based on specific assumptions on the evolution of the interest rate market, but simply follow from the absence of arbitrage. This result is extremely important since it enables us to strip, with no ambiguity, real zero-coupon bond prices from the quoted prices of zero-coupon inflation-indexed swaps.

In fact, the market quotes values of $K = K(T_M)$ for some given maturities T_M, so that equating (16.7) with the (nominal) present value of (16.1), and getting the discount factor $P_n(0, T_M)$ from the current (nominal) zero-coupon curve, we can solve for the unknown $P_r(0, T_M)$. We thus obtain the discount factor for maturity T_M in the real economy:[1]

$$P_r(0, T_M) = P_n(0, T_M)(1 + K(T_M))^M. \tag{16.8}$$

Remark 16.1.1. **(ZCIIS and Forward CPI)**. Kazziha (1999) defines the T-forward CPI at time t as the fixed amount X to be exchanged at time T for the CPI $I(T)$, for which such a swap has zero value at time t, in analogy with the definition of a forward LIBOR rate we gave in Chapter 1. From formula (16.5), we immediately obtain

$$I(t)P_r(t, T) = XP_n(t, T).$$

This is consistent with definition (15.1), which was directly based on the foreign-currency analogy.

The advantage of Kazziha's approach is that no foreign-currency analogy is required for the definition of the forward CPI's $\mathcal{I}_i$, and the pricing system she defines is only based on nominal zero-coupon bonds and forward CPI's. In her setting, the value at time zero of a T_M-forward CPI can be obtained from the market quote $K(T_M)$ by applying this simple formula

$$\mathcal{I}_M(0) = I(0)(1 + K(T_M))^M,$$

which is perfectly equivalent to (16.8).

16.2 Pricing of a YYIIS

Compared to that of a ZCIIS, the valuation of a YYIIS is more involved. Notice, in fact, that the value at time $t < T_i$ of the payoff (16.3) at time T_i is

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u)\,du} \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right] \Big| \mathcal{F}_t \right\}, \tag{16.9}$$

which, assuming $t < T_{i-1}$ (otherwise we fall back to the previous case), can be calculated as

$$N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} E_n \left[e^{-\int_{T_{i-1}}^{T_i} n(u)\,du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \Big| \mathcal{F}_{T_{i-1}} \right] \Big| \mathcal{F}_t \right\}. \tag{16.10}$$

The inner expectation is nothing but $\mathbf{ZCIIS}(T_{i-1}, T_i, I(T_{i-1}), 1)$, so that we obtain

$$\begin{aligned} &N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)] \big| \mathcal{F}_t \right\} \\ &= N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} P_r(T_{i-1}, T_i) \big| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i). \end{aligned} \tag{16.11}$$

[1] The real discount factors for intermediate maturities can be inferred by taking into account the typical seasonality effects in inflation.

The last expectation can be viewed as the nominal price of a derivative paying off, in nominal units, the real zero-coupon bond price $P_r(T_{i-1}, T_i)$ at time T_{i-1}. If real rates were deterministic, then this price would simply be the present value, in nominal terms, of the forward price of the real bond. In this case, in fact, we would have:

$$\begin{aligned} E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\, du} P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right\} &= P_r(T_{i-1}, T_i) P_n(t, T_{i-1}) \\ &= \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} P_n(t, T_{i-1}). \end{aligned}$$

In practice, however, real rates are stochastic and the expected value in (16.11) is model dependent. For instance, under dynamics (15.2), the forward price of the real bond must be corrected by a factor depending on both the nominal and real interest rates volatilities and on the respective correlation. This is explained in the following.

16.3 Pricing of a YYIIS with the JY Model

Denoting by Q_n^T the nominal T-forward measure for a general maturity T and by E_n^T the associated expectation, we can write:

$$\begin{aligned} &\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) \\ &= N\psi_i P_n(t, T_{i-1}) E_n^{T_{i-1}} \left\{ P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i). \end{aligned} \tag{16.12}$$

Remembering formula (3.39) for the zero-coupon bond price in the Hull and White (1994b) model:

$$\begin{aligned} P_r(t, T) &= A_r(t, T) e^{-B_r(t,T) r(t)}, \\ B_r(t, T) &= \frac{1}{a_r} \left[1 - e^{-a_r (T-t)} \right], \\ A_r(t, T) &= \frac{P_r^M(0, T)}{P_r^M(0, t)} \exp \left\{ B_r(t, T) f_r^M(0, t) - \frac{\sigma_r^2}{4a_r} (1 - e^{-2a_r t}) B_r(t, T)^2 \right\}, \end{aligned} \tag{16.13}$$

and noting that, by the change-of-numeraire toolkit in Section 2.3, and formula (2.12) in particular, the real instantaneous rate evolves under $Q_n^{T_{i-1}}$ according to

$$dr(t) = [-\rho_{n,r}\sigma_n\sigma_r B_n(t, T_{i-1}) + \vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]\, dt + \sigma_r\, dW_r^{T_{i-1}}(t) \tag{16.14}$$

with $W_r^{T_{i-1}}$ a $Q_n^{T_{i-1}}$-Brownian motion, we have that the real bond price $P_r(T_{i-1}, T_i)$ is lognormally distributed under $Q_n^{T_{i-1}}$, since $r(T_{i-1})$ is still a normal random variable under this (nominal) forward measure. After some tedious, but straightforward, algebra we finally obtain

$$\begin{aligned}&\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N)\\&= N\psi_i P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t,T_{i-1},T_i)} - N\psi_i P_n(t, T_i),\end{aligned} \tag{16.15}$$

where

$$\begin{aligned}C(t, T_{i-1}, T_i) =& \sigma_r B_r(T_{i-1}, T_i) \Big[B_r(t, T_{i-1}) \Big(\rho_{r,I}\sigma_I - \tfrac{1}{2}\sigma_r B_r(t, T_{i-1}) \\&+ \frac{\rho_{n,r}\sigma_n}{a_n + a_r} \big(1 + a_r B_n(t, T_{i-1})\big)\Big) - \frac{\rho_{n,r}\sigma_n}{a_n + a_r} B_n(t, T_{i-1}) \Big].\end{aligned}$$

The expectation of a real zero-coupon bond price under a nominal forward measure, in the JY model, is thus equal to the current forward price of the real bond multiplied by a correction factor, which depends on the (instantaneous) volatilities of the nominal rate, the real rate and the CPI, on the (instantaneous) correlation between nominal and real rates, and on the (instantaneous) correlation between the real rate and the CPI.

The exponential of C is the correction term we mentioned above. This term accounts for the stochasticity of real rates and, indeed, vanishes for $\sigma_r = 0$.

The value at time t of the inflation-indexed leg of the swap is simply obtained by summing up the values of all floating payments. We thus get

$$\begin{aligned}\mathbf{YYIIS}(t, \mathcal{T}, \Psi, N) =& N\psi_{\iota(t)} \left[\frac{I(t)}{I(T_{\iota(t)-1})} P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)}) \right] \\&+ N \sum_{i=\iota(t)+1}^{M} \psi_i \left[P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})}\, e^{C(t,T_{i-1},T_i)} - P_n(t, T_i) \right],\end{aligned} \tag{16.16}$$

where we set $\mathcal{T} := \{T_1, \dots, T_M\}$, $\Psi := \{\psi_1, \dots, \psi_M\}$ and $\iota(t) = \min\{i : T_i > t\}$,[2] and where the first payment after time t has been priced according to (16.6). In particular at $t = 0$,

$$\begin{aligned}\mathbf{YYIIS}(0, \mathcal{T}, \Psi, N) =& N\psi_1 [P_r(0, T_1) - P_n(0, T_1)] \\&+ N \sum_{i=2}^{M} \psi_i \left[P_n(0, T_{i-1}) \frac{P_r(0, T_i)}{P_r(0, T_{i-1})}\, e^{C(0,T_{i-1},T_i)} - P_n(0, T_i) \right] \\=& N \sum_{i=1}^{M} \psi_i P_n(0, T_i) \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)}\, e^{C(0,T_{i-1},T_i)} - 1 \right].\end{aligned} \tag{16.17}$$

The advantage of using Gaussian models for nominal and real rates is clear as far as analytical tractability is concerned. However, the possibility of negative rates and the difficulty in estimating historically the real rate parameters led to alternative approaches. We now illustrate two different market models that have been proposed for alternative valuations of a YYIIS and other inflation-indexed derivatives.

[2] By definition, $T_{\iota(t)-1} \le t < T_{\iota(t)}$.

16.4 Pricing of a YYIIS with a First Market Model

For an alternative pricing of the above YYIIS, we notice that we can change measure and, as explained in Section 2.8, re-write the expectation in (16.12) as

$$\begin{aligned} P_n(t,T_{i-1})E_n^{T_{i-1}}\left\{P_r(T_{i-1},T_i)\big|\mathcal{F}_t\right\} &= P_n(t,T_i)E_n^{T_i}\left\{\frac{P_r(T_{i-1},T_i)}{P_n(T_{i-1},T_i)}\Big|\mathcal{F}_t\right\} \\ &= P_n(t,T_i)E_n^{T_i}\left\{\frac{1+\tau_i F_n(T_{i-1};T_{i-1},T_i)}{1+\tau_i F_r(T_{i-1};T_{i-1},T_i)}\Big|\mathcal{F}_t\right\}, \end{aligned} \tag{16.18}$$

which can be calculated as soon as we specify the distribution of both forward rates under the nominal T_i-forward measure.

It seems natural, therefore, to resort to a LFM, which postulates the evolution of simply-compounded forward rates, namely the variables that explicitly enter the last expectation, see Section 6.3. This approach, followed by Mercurio (2005), is detailed in the following.

Since $I(t)P_r(t,T_i)$ is the price of an asset in the nominal economy, we have that the forward CPI

$$\mathcal{I}_i(t) = I(t)\frac{P_r(t,T_i)}{P_n(t,T_i)}$$

is a martingale under $Q_n^{T_i}$ by the definition itself of $Q_n^{T_i}$. Assuming lognormal dynamics for $\mathcal{I}_i$,

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\,dW_i^I(t), \tag{16.19}$$

where $\sigma_{I,i}$ is a positive constant and W_i^I is a $Q_n^{T_i}$-Brownian motion, and assuming also that both nominal and real forward rates follow a LFM, the analogy with cross-currency derivatives pricing implies that the dynamics of $F_n(\cdot;T_{i-1},T_i)$ and $F_r(\cdot;T_{i-1},T_i)$ under $Q_n^{T_i}$ are given by (see Section 14.4)

$$\begin{aligned} dF_n(t;T_{i-1},T_i) &= \sigma_{n,i}F_n(t;T_{i-1},T_i)\,dW_i^n(t), \\ dF_r(t;T_{i-1},T_i) &= F_r(t;T_{i-1},T_i)\big[-\rho_{I,r,i}\sigma_{I,i}\sigma_{r,i}\,dt + \sigma_{r,i}\,dW_i^r(t)\big], \end{aligned} \tag{16.20}$$

where $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants, W_i^n and W_i^r are two Brownian motions with instantaneous correlation ρ_i, and $\rho_{I,r,i}$ is the instantaneous correlation between $\mathcal{I}_i(\cdot)$ and $F_r(\cdot;T_{i-1},T_i)$, i.e. $dW_i^I(t)\,dW_i^r(t) = \rho_{I,r,i}\,dt$.

Allowing $\sigma_{I,i}$, $\sigma_{n,i}$ and $\sigma_{r,i}$ to be deterministic functions of time does not complicate the calculations below. We assume hereafter that such volatilities are constant for ease of notation only. In practice, however, the implications of using constant or time-dependent coefficients should be carefully analyzed. See also Chapter 7 and Remark 18.0.1 below.

The expectation in (16.18) can then be easily calculated with a numerical integration by noting that, under $Q_n^{T_i}$ and conditional on $\mathcal{F}_t$, the pair[3]

[3] To lighten the notation, we simply write (X_i, Y_i) instead of $(X_i(t), Y_i(t))$.

$$(X_i, Y_i) = \left(\ln \frac{F_n(T_{i-1}; T_{i-1}, T_i)}{F_n(t; T_{i-1}, T_i)}, \ln \frac{F_r(T_{i-1}; T_{i-1}, T_i)}{F_r(t; T_{i-1}, T_i)} \right) \tag{16.21}$$

is distributed as a bivariate normal random variable with mean vector and variance-covariance matrix, respectively, given by

$$M_{X_i,Y_i} = \begin{bmatrix} \mu_{x,i}(t) \\ \mu_{y,i}(t) \end{bmatrix}, \quad V_{X_i,Y_i} = \begin{bmatrix} \sigma_{x,i}^2(t) & \rho_i \sigma_{x,i}(t)\sigma_{y,i}(t) \\ \rho_i \sigma_{x,i}(t)\sigma_{y,i}(t) & \sigma_{y,i}^2(t) \end{bmatrix}, \tag{16.22}$$

where

$$\mu_{x,i}(t) = -\tfrac{1}{2}\sigma_{n,i}^2 (T_{i-1} - t), \quad \sigma_{x,i}(t) = \sigma_{n,i}\sqrt{T_{i-1} - t},$$
$$\mu_{y,i}(t) = \left[-\tfrac{1}{2}\sigma_{r,i}^2 - \rho_{I,r,i}\sigma_{I,i}\sigma_{r,i} \right](T_{i-1} - t), \quad \sigma_{y,i}(t) = \sigma_{r,i}\sqrt{T_{i-1} - t}.$$

It is well known that the density $f_{X_i,Y_i}(x, y)$ of (X_i, Y_i) can be decomposed as[4]

$$f_{X_i,Y_i}(x, y) = f_{X_i|Y_i}(x, y) f_{Y_i}(y),$$

where

$$f_{X_i|Y_i}(x, y) = \frac{1}{\sigma_{x,i}(t)\sqrt{2\pi}\sqrt{1-\rho_i^2}} \exp\left[-\frac{\left(\frac{x-\mu_{x,i}(t)}{\sigma_{x,i}(t)} - \rho_i \frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)} \right)^2}{2(1-\rho_i^2)} \right]$$
$$f_{Y_i}(y) = \frac{1}{\sigma_{y,i}(t)\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{y - \mu_{y,i}(t)}{\sigma_{y,i}(t)} \right)^2 \right]. \tag{16.23}$$

The last expectation in (16.18) can thus be calculated as

$$\int_{-\infty}^{+\infty} \frac{\int_{-\infty}^{+\infty} (1 + \tau_i F_n(t; T_{i-1}, T_i) e^x) f_{X_i|Y_i}(x, y)\, dx}{1 + \tau_i F_r(t; T_{i-1}, T_i)\, e^y} f_{Y_i}(y)\, dy$$
$$= \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(t; T_{i-1}, T_i)\, e^{\mu_{x,i}(t) + \rho_i \sigma_{x,i}(t) \frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)} + \frac{1}{2}\sigma_{x,i}^2(t)(1-\rho_i^2)}}{1 + \tau_i F_r(t; T_{i-1}, T_i)\, e^y} f_{Y_i}(y)\, dy$$
$$= \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(t; T_{i-1}, T_i)\, e^{\rho_i \sigma_{x,i}(t) z - \frac{1}{2}\sigma_{x,i}^2(t)\rho_i^2}}{1 + \tau_i F_r(t; T_{i-1}, T_i)\, e^{\mu_{y,i}(t) + \sigma_{y,i}(t) z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz,$$

yielding:

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N)$$
$$= N\psi_i P_n(t, T_i) \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(t; T_{i-1}, T_i)\, e^{\rho_i \sigma_{x,i}(t) z - \frac{1}{2}\sigma_{x,i}^2(t)\rho_i^2}}{1 + \tau_i F_r(t; T_{i-1}, T_i)\, e^{\mu_{y,i}(t) + \sigma_{y,i}(t) z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$
$$- N\psi_i P_n(t, T_i). \tag{16.24}$$

[4] See also Appendix E for a similar calculation.

To value the whole inflation-indexed leg of the swap some care is needed, since we cannot simply sum up the values (16.24) of the single floating payments. In fact, as noted by Schlögl (2002) in a multi-currency version of the LFM,[5] we cannot assume that the volatilities $\sigma_{I,i}$, $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants for all i, because there exists a precise relation between two consecutive forward CPIs and the corresponding nominal and real forward rates, namely:

$$\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} = \frac{1+\tau_i F_n(t;T_{i-1},T_i)}{1+\tau_i F_r(t;T_{i-1},T_i)}. \tag{16.25}$$

Clearly, if we assume that $\sigma_{I,i}$, $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants, $\sigma_{I,i-1}$ cannot be constant as well, and its admissible values are obtained by equating the (instantaneous) quadratic variations on both sides of (16.25).

However, by freezing the forward rates at their time 0 value in the diffusion coefficients of the right-hand-side of (16.25), we can still get forward CPI volatilities that are approximately constant. For instance, in the one-factor model case,

$$\begin{aligned}\sigma_{I,i-1} &= \sigma_{I,i} + \sigma_{r,i}\frac{\tau_i F_r(t;T_{i-1},T_i)}{1+\tau_i F_r(t;T_{i-1},T_i)} - \sigma_{n,i}\frac{\tau_i F_n(t;T_{i-1},T_i)}{1+\tau_i F_n(t;T_{i-1},T_i)}\\ &\approx \sigma_{I,i} + \sigma_{r,i}\frac{\tau_i F_r(0;T_{i-1},T_i)}{1+\tau_i F_r(0;T_{i-1},T_i)} - \sigma_{n,i}\frac{\tau_i F_n(0;T_{i-1},T_i)}{1+\tau_i F_n(0;T_{i-1},T_i)}.\end{aligned}$$

Therefore, applying this "freezing" procedure for each $i < M$ starting from $\sigma_{I,M}$, or equivalently for each $i > 2$ starting from $\sigma_{I,1}$, we can still assume that the volatilities $\sigma_{I,i}$ are all constant and set to one of their admissible values. The value at time t of the inflation-indexed leg of the swap is thus given by

$$\begin{aligned}&\mathbf{YYIIS}(t,\mathcal{T},\Psi,N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right]\\ &\quad + N\sum_{i=\iota(t)+1}^{M}\psi_i P_n(t,T_i)\\ &\qquad \cdot\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(t;T_{i-1},T_i)\, e^{\rho_i\sigma_{x,i}(t)z-\frac{1}{2}\sigma_{x,i}^2(t)\rho_i^2}}{1+\tau_i F_r(t;T_{i-1},T_i)\, e^{\mu_{y,i}(t)+\sigma_{y,i}(t)z}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz - 1\right].\end{aligned} \tag{16.26}$$

In particular at $t = 0$,

[5] See also Section 14.5.4.

$$\mathbf{YYIIS}(0,\mathcal{T},\Psi,N) = N\psi_1[P_r(0,T_1) - P_n(0,T_1)] + N\sum_{i=2}^{M}\psi_i P_n(0,T_i)$$
$$\cdot\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(0;T_{i-1},T_i)\,e^{\rho_i\sigma_{x,i}(0)z-\frac{1}{2}\sigma_{x,i}^2(0)\rho_i^2}}{1+\tau_i F_r(0;T_{i-1},T_i)\,e^{\mu_{y,i}(0)+\sigma_{y,i}(0)z}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz - 1\right]$$
$$= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)$$
$$\cdot\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(0;T_{i-1},T_i)\,e^{\rho_i\sigma_{x,i}(0)z-\frac{1}{2}\sigma_{x,i}^2(0)\rho_i^2}}{1+\tau_i F_r(0;T_{i-1},T_i)\,e^{\mu_{y,i}(0)+\sigma_{y,i}(0)z}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz - 1\right]. \tag{16.27}$$

This YYIIS price depends on the following parameters: the (instantaneous) volatilities of nominal and real forward rates and their correlations, for each payment time T_i, $i = 2,\dots,M$; the (instantaneous) volatilities of forward inflation indices and their correlations with real forward rates, again for each $i = 2,\dots,M$.

Compared with expression (16.17), formula (16.27) looks more complicated both in terms of input parameters and in terms of the calculations involved. However, one-dimensional numerical integrations are not so cumbersome and time consuming. Moreover, as is typical in a market model, the input parameters can be determined more easily than those coming from the previous short-rate approach. In this respect, formula (16.27) is preferable to (16.17).

As in the JY case, valuing a YYIIS with a LFM has the drawback that the volatility of real rates may be hard to estimate, especially when resorting to a historical calibration. This is why, in the literature, a second market model has been proposed, which enables us to overcome this estimation issue. In the following section we will review this approach, which has been independently developed by Kazziha (1999), Belgrade, Benhamou and Koehler (2004) and Mercurio (2005).

16.5 Pricing of a YYIIS with a Second Market Model

Applying the definition of forward CPI and using the fact that $\mathcal{I}_i$ is a martingale under $Q_n^{T_i}$, we can also write, for $t < T_{i-1}$,

$$\begin{aligned}\mathbf{YYIIS}(t,T_{i-1},T_i,\psi_i,N) &= N\psi_i P(t,T_i)E_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})} - 1\middle|\mathcal{F}_t\right\}\\ &= N\psi_i P(t,T_i)E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - 1\middle|\mathcal{F}_t\right\}\\ &= N\psi_i P(t,T_i)E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} - 1\middle|\mathcal{F}_t\right\}.\end{aligned} \tag{16.28}$$

The dynamics of $\mathcal{I}_i$ under $Q_n^{T_i}$ is given by (16.19) and an analogous evolution holds for $\mathcal{I}_{i-1}$ under $Q_n^{T_{i-1}}$. The dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$ can be derived by applying the change-of-numeraire toolkit in Section 2.3. We get:

$$d\mathcal{I}_{i-1}(t) = \mathcal{I}_{i-1}(t)\sigma_{I,i-1}\left[-\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_iF_n(t;T_{i-1},T_i)}\rho_{I,n,i}\,dt + dW_{i-1}^I(t)\right], \tag{16.29}$$

where $\sigma_{I,i-1}$ is a positive constant, W_{i-1}^I is a $Q_n^{T_i}$-Brownian motion with $dW_{i-1}^I(t)\,dW_i^I(t) = \rho_{I,i}\,dt$, and $\rho_{I,n,i}$ is the instantaneous correlation between $\mathcal{I}_{i-1}(\cdot)$ and $F_n(\cdot;T_{i-1},T_i)$.

The evolution of $\mathcal{I}_{i-1}$, under $Q_n^{T_i}$, depends on the nominal forward rate $F_n(\cdot;T_{i-1},T_i)$, so that the calculation of (16.28) is rather involved in general. To avoid unpleasant complications, like those induced by higher-dimensional integrations, we freeze the drift in (16.29) at its current time-t value, so that $\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ is lognormally distributed also under $Q_n^{T_i}$. This leads to

$$E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})}|\mathcal{F}_t\right\} = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)},$$

where

$$D_i(t) = \sigma_{I,i-1}\left[\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_iF_n(t;T_{i-1},T_i)}\rho_{I,n,i} - \rho_{I,i}\sigma_{I,i} + \sigma_{I,i-1}\right](T_{i-1}-t),$$

so that

$$\begin{aligned}\mathbf{YYIIS}(t,T_{i-1},T_i,\psi_i,N) &= N\psi_iP_n(t,T_i)\left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)} - 1\right]\\ &= N\psi_iP_n(t,T_i)\left[\frac{P_n(t,T_{i-1})P_r(t,T_i)}{P_n(t,T_i)P_r(t,T_{i-1})}\,e^{D_i(t)} - 1\right].\end{aligned} \tag{16.30}$$

Finally, the value at time t of the inflation-indexed leg of the swap is

$$\begin{aligned}\mathbf{YYIIS}(t,\mathcal{T},\Psi,N) &= N\psi_{\iota(t)}P_n(t,T_{\iota(t)})\left[\frac{\mathcal{I}_{\iota(t)}(t)}{I(T_{\iota(t)-1})} - 1\right]\\ &\quad + N\sum_{i=\iota(t)+1}^{M}\psi_iP_n(t,T_i)\left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)} - 1\right]\\ &= N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right]\\ &\quad + N\sum_{i=\iota(t)+1}^{M}\psi_i\left[P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{D_i(t)} - P_n(t,T_i)\right].\end{aligned} \tag{16.31}$$

In particular at $t=0$,

$$\begin{aligned}
\mathbf{YYIIS}(0,\mathcal{T},\Psi,N) &= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\frac{\mathcal{I}_i(0)}{\mathcal{I}_{i-1}(0)}\,e^{D_i(0)}-1\right]\\
&= N\psi_1[P_r(0,T_1)-P_n(0,T_1)]\\
&\quad + N\sum_{i=2}^{M}\psi_i\left[P_n(0,T_{i-1})\frac{P_r(0,T_i)}{P_r(0,T_{i-1})}\,e^{D_i(0)}-P_n(0,T_i)\right]\\
&= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\frac{1+\tau_i F_n(0;T_{i-1},T_i)}{1+\tau_i F_r(0;T_{i-1},T_i)}\,e^{D_i(0)}-1\right].
\end{aligned} \tag{16.32}$$

This YYIIS price depends on the following parameters: the (instantaneous) volatilities of forward inflation indices and their correlations; the (instantaneous) volatilities of nominal forward rates; the instantaneous correlations between forward inflation indices and nominal forward rates.

Expression (16.32) looks pretty similar to (16.17) and may be preferred to (16.27) since it combines the advantage of a fully-analytical formula with that of a market-model approach. Moreover, contrary to (16.27), the correction term D does not depend on the volatility of real rates.

A drawback of formula (16.32) is that the approximation it is based on may be rough for long maturities T_i. In fact, such a formula is exact when the correlations $\rho_{I,n,i}$ are set to zero and the terms D_i are simplified accordingly. In general, however, such correlations can have a non-negligible impact on the D_i, and non-zero values can be found when calibrating the model to YYIIS market data.

To visualize the magnitude of the correction terms D_i in the pricing formula (16.32), we plot in Figure 16.2 the values of $D_i(0)$ corresponding to setting $T_i = i$ years, $i = 2, 3, \dots, 20$, $\sigma_{I,i} = 0.006$, $\sigma_{n,i} = 0.22$, $\rho_{I,n,i} = 0.2$, $\rho_{I,i} = 0.6$, for each i, and where the forward rates $F_n(0; T_{i-1}, T_i)$ are stripped from the Euro nominal zero-coupon curve as of 7 October 2004.

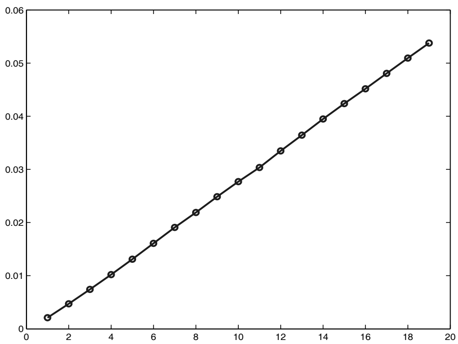

Fig. 16.2. Plot of values $D_i(0)$, in percentage points, for $i = 2, 3, \dots, 20$.

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